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Approximation of Curves and Surfaces by Algebraic Curves and Surfaces

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Approximation of Curves and Surfaces by Algebraic Curves and Surfaces

A DISSERTATION

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BY

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P. A. SMITH

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By P. A. Smith.

1. Let J be a simple closed curve lying in the (x, y)-plane. By the Theorem of Jordan, J separates the plane into two regions, — the interior and exterior regions. The object of this paper is to obtain a polynomial series

$$\sum_{n=1}^{\infty} A_n(x, y) \tag{1}$$

which will converge uniformly in a region R containing J to a continuous function which is 0 on J, and which, in R, is > 0 at points exterior to J and < 0 at interior points. The series (1) moreover is to give rise (by equating successive sums to zero) to a sequence of *non-singular algebraic ovals* converging to J in a manner explicitely described in Theorem 1.

Analogous results will be obtained for $(n-1)$ dimensional manifolds in n-space (for example, simple closed surfaces in 3-space) but only for a restricted class of such manifolds (§ 7). The approximation of the most general manifolds is discussed in § 8.

A special type of approximation of curves, namely by ovals of *unicursal* algebraic curves, is treated in § 5. It will not follow however that the series (1) can be chosen so as to give rise to a sequence of unicursal curves (see § 6).

Let $J(x, y)$ be a "distance function" with respect to J defined as follows: if $\delta(x, y)$ is the shortest distance from the point (x, y) to J, then $J(x, y) = \delta(x, y)$ for (x, y) exterior to J, $J(x, y) = -\delta(x, y)$ interior to J, and $J(x, y) = 0$ for (x, y) on J. It is evident that $J(x, y)$ is bounded and continuous in any closed region of the plane.

Let R be the rectangular region $a < x < b$, $c < y < d$, where a, b, c, d are so chosen that J will lie completely in R, and let R' designate R plus its boundary.* The existence of the series (1) will follow from

Theorem 1. *Let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers converging to zero. There exists for each n a polynomial $P_n(x, y)$ such that*

$$| P_n(x, y) - J(x, y) | < \varepsilon_n$$

* The accent will have this significance throughout this paper when the accented letter denotes a region.

in R', while the real part of the curve $P_n(x, y) = 0$ in R consists of a non-singular oval in (1,1) *continuous correspondence with J, the distance between corresponding points being $< \varepsilon_n$.*

This last situation will be described more briefly by the statement: "The oval is mapped (ε_n) on J".

For the series (1) we may obviously take

$$P_1(x, y) + \sum_{n=1}^{\infty} (P_{n+1}(x, y) - P_n(x, y))$$

since the sum of the first n terms of this series is $P_n(x, y)$.*

2. **Polynomial approximation.** We shall make use of the following approximation theorem:

Let S be the region defined by $c_1 < x < c_2$, $d_1 < y < d_2$ and let $f(x, y)$ be a function defined and continuous at every point of S' and such that $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous at every point of T' where T is a region which with its boundary lies in S. Then there exists a sequence of polynomials $\varphi_n(x, y)$ which converges uniformly to $f(x, y)$ in S' while $\partial \varphi_n/\partial y$ and $\partial \varphi_n/\partial y$ converge uniformly to $\partial f/\partial x$ and $\partial f/\partial y$ respectively in T'.

A proof of this theorem for one variable using for $\varphi_n(x)$ the polynomial

$$\frac{k_n}{2} \int_0^1 f(u) [1 - (u - x)^2]^n \, du, \qquad k_n = \frac{3 \cdot 5 \cdots (2n + 1)}{2 \cdot 4 \cdots 2n}$$

is given by de la Vallée-Poussin.† The extension of this method to two (or n) variables using for φ_n the polynomial

$$\left(\frac{k_n}{2}\right)^2 \int_0^1 \int_0^1 f(u, v) [1 - (u - x)^2]^n [1 - (v - y)^2]^n \, du \, dv$$

is automatic. It is assumed for this purpose that $0 < c_1 < c_2 < 1$ and $0 < d_1 < d_2 < 1$, which is permissable since S may be transformed by a suitable linear transformation into a region lying inside the square $0 < x, y < 1$. We further assume that the region of definition of $f(x, y)$ has been extended in such a way that $f(x, y)$ is defined and continuous not only in S but at every point of the region $0 \leqq x, y \leqq 1$.‡

* By known theorems on polynomial approximation (see for example Borel, Leçons sur les Fonctions de Variables Réelles, p. 62) there exist polynomials $Q_n(x, y)$ converging to $J(x, y)$. But in general the real branches of $Q_n = 0$ will have complicated singularities and hence Theorem 1 is not a trivial application of polynomial approximation theory.

† Bull. de l'Acad. Belg., Cl. des Sci., 1908, p. 193. Landau (Rend. del Circ. Mat. di Palermo, vol. 25, 1908, p. 337) uses polynomials similarly defined and proves their convergence for n variables but does not treat the convergence of the derivatives.

‡ We may define $f(x, y)$ outside of S' as follows: let Z be a fixed point of S and let an arbitrary ray having Z for an end-point cut the boundary of S at the point Z_1, (x_1, y_1)

3. **Proof of Theorem 1.** We shall prove this theorem by establishing a sequence of propositions which we shall mark with Roman numerals for reference.

I. *Let η_1 be an arbitrary positive number. There can be inscribed in J a simple polygon Π which can be mapped (η_1) on J.*

Proof. Let a parametric representation of J be given by

$$x = f(t), \qquad y = g(t)$$

where f and g are continuous for all finite values of t and each possesses the period ω. The distance between two points of J whose parameter values are t_1 and t_2 is

$$+\sqrt{[f(t_1)-f(t_2)]^2+[g(t_1)-g(t_2)]^2}$$

and, since f and g are continuous, we can choose a number $\delta > 0$ sufficiently small such that, if any two t's differ by less than δ, the distance between the corresponding points of J is $< \eta_1/2$.

Let us divide the interval $(t_0, t_0 + \omega)$ into a finite number of sub-intervals each of length $< \delta/2$. J is correspondingly divided into the arcs $\alpha_1, \alpha_2, \cdots, \alpha_p$ named in circular order and two points on any α or on two successive α's are within a distance of $\eta_1/2$ from each other. We can now inscribe in J a simple polygon Π in such a way that at least one vertex of Π shall fall on each α and such that the vertices of Π shall follow each other in the same order with respect to Π as they do with respect to J.*

Let $V_1, V_2, \cdots, V_k, V_{k+1} = V_1$ be the vertices of Π named in the order of t increasing and call e_i the edge $V_i V_{i+1}$ $(i = 1, 2, \cdots, k)$. As t increases continuously from t_{ν_1}† to t_{ν_2}, an arc β_1 is described on J. The points V_1

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and the boundary of the region $0 < x, y < 1$ at the point Z_2; then define $f(x, y) = f(x_1, y_1)$ on the segment $Z_1 Z_2$.

The method of de la Vallée-Poussin leads to more general results than those here stated. Granting these results for the convergence of the $\varphi_n(x, y)$ and making certain assumptions concerning the derivatives of $f(x, y)$, the convergence of the derivatives of the $\varphi_n(x, y)$ can be proved very simply. We define $f(x, y)$ to be zero for points not in S'. Then on integrating $\dfrac{\partial \varphi_n}{\partial x}$ by parts, we obtain a polynomial of the form φ_n with $f(u, v)$ replaced by $\dfrac{\partial f(u, v)}{\partial u}$.

* A proof of this statement is given by de la Vallée-Poussin in his Cours d'Analyse Infinitésimale, 3rd ed., vol. 1, p. 379. The first step in the proof is to divide the plane into small rectangular regions q_i by a net of straight lines. There is a slight ambiguity in the proof in case any vertex of a q_i falls on J. This may be avoided however by slightly altering the net in the neighborhood of such vertices.

† t_{ν_1} being the value of t corresponding to V_1 in the interval $(t_0, t_0 + \omega)$.

and V_2 are on the same α or on two successive α's. In either case the distance V_1V_2 is smaller than $\eta_1/2$. Furthermore, if C (fig. 1) is any point of β_1, the distance V_1C is $<\eta_1/2$, since $|t_{V_1}-t_C|<\delta/2$. Hence each point of e_1 is within a distance η_1 from every point of β_1. Hence e_1' ($=e_1$ plus its end-points) can be mapped (η_1) on β_1', and this can be done in such a way that V_1 and V_2, considered as end-points of β_1, correspond respectively to themselves considered as end-points of e_1.

The remaining edges $e_2, \cdots, e_k$ are now mapped in the same way on the corresponding arcs $\beta_2, \cdots, \beta_k$. Thus Π has been mapped (η_1) on J and the proof of I completed.

Consider now the closed curve $\sigma_i = e_i + \beta_i + V_i + V_{i+1}$ which may have multiple points. Let O be a point of the plane lying on neither Π nor J

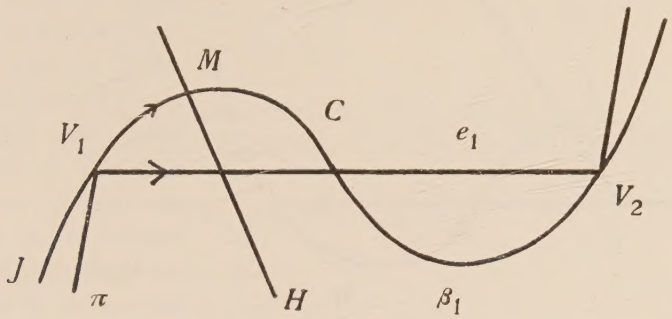

Fig. 1.

and let a ray be pivoted on O and pass through a movable point M of σ_i. As M describes σ_i once, the total angular variation of the ray is either zero or a non-zero multiple of 2π.

Let L be the set of points of the plane which lie on neither Π nor J and let $C_i (i = 1, \cdots, k)$ consist of those points of L for which the total angular variation with regard to σ_i is a non-zero multiple of 2π. Let C be the logical sum $C_1 + \cdots + C_k$ and D the set $L - C$.*

II. *Each point of D is either interior to both J and Π or exterior to both.*

For, let H be a point of D. Call v_J^1 the angular variation of a ray pivoting on H as the movable pointpasses from V_1 to V_2 along β_1 and call v_π^1 the corresponding variation when the movable point describes e_1 instead of β_1 (fig. 1).

The total angular variation with respect to H as the movable point completely describes σ_1 is evidently $v_J^1 - v_\pi^1$. But this variation is zero since H is not in C_1. Hence $v_J^1 = v_\pi^1$. Similarly $v_J^2 = v_\pi^2$, and so on.

* This set is not null; it contains for example the points of the plane not in R'.

Therefore the variation with respect to H as the movable point completely describes Π is exactly equal to the variation as the point describes J. Now, the total variation with respect to H as the movable point describes any simple closed curve not on H is either 0 or 2π according as H is exterior or interior to that curve.* Hence H is either exterior to both Π and J, or interior to both.

III. *Each point N of the set C is within a distance η_1 from Π and from J.*

For N is a point of some C_i, say C_1. Let E be a point of e_1. Draw the linear segment EN and extend it indefinitely beyond N forming the ray $EN\infty$ (fig. 2). The ray $N\infty$ must meet β_1. For if not, $N\infty$ does not touch σ_1 at all and hence there would be a position which a ray pivoting on N never assumes as the movable point describes σ_1; and this is absurd since the angular variation with respect to any point of C_1 as the movable point describes σ_1 is a non-zero multiple of 2π. Call, then, F a point which $N\infty$ has in common with β_1. Then $EF<\eta_1$ by the construction of Π and hence $EN<\eta_1$ and $NF<\eta_1$. This establishes III.

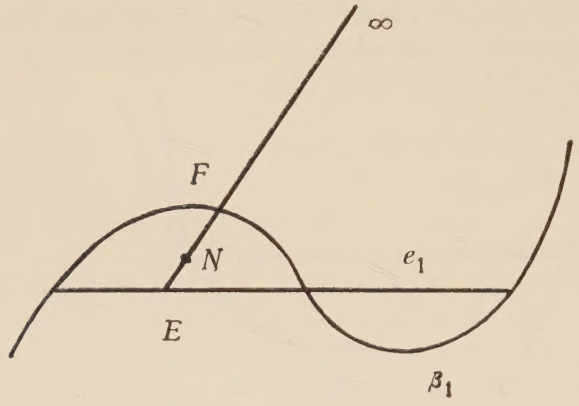

Fig. 2.

Let $\Pi(x, y)$ be the distance function with respect to Π. Then

IV. *The inequality*

$$|\Pi(x, y) - J(x, y)| < 2\eta_1$$

holds for all finite points of the plane.

For if (x, y) is on both Π and J, then $\Pi(x, y) = J(x, y) = 0$. Next, if (x, y) is interior to one of Π or J and exterior to the other, then it belongs to the set C and hence by III

$$|\Pi(x, y)| < \eta_1, \qquad |J(x, y)| < \eta_1.$$

Finally, suppose that (x, y) is interior to both Π and J or exterior to both. Then $\Pi(x, y)$ and $J(x, y)$ are of like sign and their absolute values differ by less than η_1. For since each point of Π is within a distance of η_1

* See Kerékjártó, Vorlesungen über Topologie I, p. 84.

from J and conversely, the shortest distance from (x, y) to J can not differ from the shortest distance to Π by more than η_1.

We now construct a "rounded-off" polygon K by fitting arcs of tangent circles into the corners of Π as shown in fig. 3.

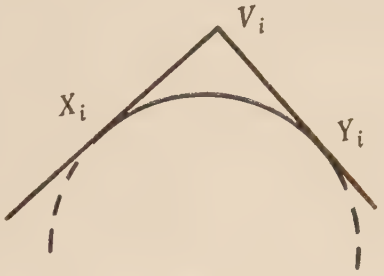

Fig. 3.

Let η_2 be an arbitrary positive number. Manifestly, if the circles used in constructing K are sufficiently small, then (a) K will be free from multiple points, and (b) K can be mapped (η_2) on Π. For (b) it is necessary to take the circles so small that each arc $X_i Y_i$ (fig. 3) can be mapped (η_2) on the corresponding broken line $X_i V_i Y_i$. By virtue of I we have

V. *K can be mapped $(\eta_1 + \eta_2)$ on J.*

Let $K(x, y)$ be the distance function with respect to K. By applying the reasoning in the proofs of III and IV to the curves Π and K, we obtain

$$|\Pi(x, y) - K(x, y)| < 2\eta_2.$$

Combining this with IV, we have

VI. *For all finite points of the plane*

$$|K(x, y) - J(x, y)| < 2(\eta_1 + \eta_2). \tag{2}$$

K consists alternately of linear segments and circular arcs, together with all the end-points. Referring alike to linear segments and arcs as edges, each edge to include its end-points, let p be the shortest distance between any two non-consecutive edges of K. Let q be the length of the smallest radius of the circles used in constructing K, and let $\mu > 0$ be smaller than each of $p/2$, q.

Now let P be an arbitrary point of K and s a linear segment of length 2μ, normal to K and having P for its mid-point. When P is fixed, we shall call s together with its end-points a *normal segment.*

VII. *Two normal segments having distinct mid-points can not meet.*

This is obvious if the two mid-points P_1 and P_2 are on the same linear segment of K. If they are on the same arc or on a linear segment and its adjacent are, VII holds since $\mu < q$. Suppose then that P_1 and P_2 are on two non-consecutive edges of K. Then if the corresponding normal segments meet, the distance between P_1 and P_2 would be $\leqq 2\mu < p$, contrary to the definition of p.

VIII. *A normal segment with mid-point P has no other point in common with K.*

This is a corollary of VII.

We now let P describe a complete circuit on K. It follows* from VIII that one end-point of s remains interior to K and the other exterior. Hence the end-points of s trace two closed curves,—one, Γ_e lying wholly exterior to K and the other, Γ_i, wholly interior. Since $\mu < q$, Γ_e and Γ_i will have no corners. They will in fact, consist of linear segments parallel to those of K and circular arcs concentric with the arcs of K (fig. 4).

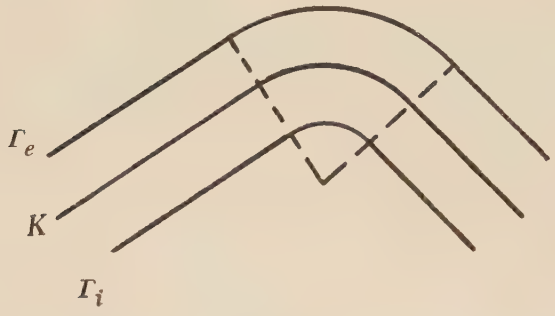

Fig. 4.

IX. *Γ_e and Γ_i possess no multiple points.*

For a multiple point of either curve would be on two or more normal segments through distinct points of K, contrary to VII.

The totality of normal segments constitutes a ring-shaped region U containing K and having the boundary $\Gamma_e + \Gamma_i$.

X. *Each point of U' is on one and only one normal segment.*

Each point of U is on at least one by construction of U. That it can be on only one follows from VII.

We now consider the distance function $K(x, y)$. From the nature of K and of the region U we see that the surface $z = K(x, y)$ will consist, for (x, y) in U of pieces of planes and pieces of right circular cones joined alternately,—the planes and the elements of the cones making uniformly an angle of 45° with the (x, y)-plane. A normal to the surface at the point (x, y, z) would vary continuously as (x, y) moves continuously in U; that is, its direction cosines are continuous functions of x and y in U. Hence $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial y}$ exist and are continuous at each point of U.

Let $\eta_3 > 0$ be arbitrarily small and let $\bar{\mu} > 0$ be smaller than each of μ, η_3. A region $\bar{U}$ is now to be constructed precisely as we constructed

* It is a theorem proved by Lennes (Theorems on Simple Finite Polygon and Polyhedron, Am. Journ., 1911, p. 43) that, if a linear segment AB meets a polygon in only one point C not a vertex, then AC and BC lie one in the interior region of the polygon and one in the exterior region. It is a simple matter to extend this theorem to a polygon such as K, with C any point of the polygon and AB not tangent to one of the arcs.

U except that we shall now take s to be of length $2\bar{\mu}$. Let $\bar{\Gamma}_e$ and $\bar{\Gamma}_i$ be the new curves corresponding to the curves Γ_e and Γ_i. $\bar{U}$ together with its boundary lies wholly in U. Hence $\frac{\partial K}{\partial x}$ and $\frac{\partial K}{\partial y}$ are continuous at each point of $\bar{U}$ *and its boundary*. When (x, y) is on $\bar{\Gamma}_e$, $K(x, y) = \bar{\mu}$, while for (x, y) on $\bar{\Gamma}_i$, $K(x, y) = -\bar{\mu}$. Moreover

$$\begin{aligned} &K(x, y) > \bar{\mu} \qquad \text{for } (x, y) \text{ exterior to } \bar{\Gamma}_e, \\ &K(x, y) < \bar{\mu} \qquad \text{for } (x, y) \text{ interior to } \bar{\Gamma}_i. \end{aligned} \tag{3}$$

By the theorem of § 2 we can obtain now a polynomial $P(x, y)$ such that in R'*

$$|P(x, y) - K(x, y)| < \bar{\mu} \tag{4}$$

while in $\bar{U}'$,

$$\left|\frac{\partial P}{\partial x} - \frac{\partial K}{\partial x}\right| < \frac{1}{2}, \qquad \left|\frac{\partial P}{\partial y} - \frac{\partial K}{\partial y}\right| < \frac{1}{2}. \tag{4a}$$

By (2) and (4) we shall have, since $\bar{\mu} < \eta_3$,

$$|P(x, y) - J(x, y)| < 2(\eta_1 + \eta_2) + \eta_3 \tag{5}$$

uniformly in R'.

By virtue of (3) and (4) for (x, y) in R'

$$\begin{aligned} &P(x, y) > 0 \qquad \text{on and exterior to } \bar{\Gamma}_e, \\ &P(x, y) < 0 \qquad \text{on and interior to } \bar{\Gamma}_i. \end{aligned}$$

Hence the points of R' at which $P(x, y)$ is zero lie in $\bar{U}$. These points constitute a set Δ of one or more real branches of the curve $P(x, y) = 0$.

XI. *Δ meets each normal segment of $\bar{U}$ at one and only one point.*

Proof. Each normal segment intersects Δ at least once; for $P(x, y) > 0$ at that end of the segment which lies on $\bar{\Gamma}_e$ while $P(x, y) < 0$ at the other end.

Each normal segment meets Δ at most once. To prove this we note first that

$$\frac{dP}{dn} - \frac{dK}{dn} = \left(\frac{\partial P}{\partial x} - \frac{\partial K}{\partial x}\right) \cos\alpha + \left(\frac{\partial P}{\partial y} - \frac{\partial K}{\partial y}\right) \sin\alpha$$

where $\frac{dP}{dn}$ and $\frac{dK}{dn}$ denote derivatives along a normal to K in an out-

* We can obviously assume that R contains all the geometric elements which we have constructed up to this point, in particular, Γ_e and Γ_i. For if this were not the case, we could here increase the size of R.

ward direction, α being the angle which the normal makes with the x-axis. Hence

$$\left|\frac{dP}{dn}-\frac{dK}{dn}\right| \leqq \left|\frac{\partial P}{\partial x}-\frac{\partial K}{\partial x}\right| |\cos\alpha| + \left|\frac{\partial P}{\partial y}-\frac{\partial K}{\partial y}\right| |\sin\alpha|$$

and hence by (4a)

$$\left|\frac{dP}{dn}-\frac{dK}{dn}\right| < \frac{1}{2}(|\sin\alpha|+|\cos\alpha|) < 1. \tag{6}$$

Now if $P(x, y)$ is zero at two distinct points S_1 and S_2 of a normal segment, then at a third point S_3 lying on the segment $S_1 S_2$ we would have $\frac{dP}{dn} = 0$ by Rolle's Theorem. Then at S_3

$$\left|\frac{dP}{dn}-\frac{dK}{dn}\right| = 1$$

since $\frac{dK}{dn} = 1$ at every point of $\bar{U}$. This contradicts (6) and hence XI is proved.

XII. *Δ can be mapped (η_3) on K.*

Proof. By making each point P of K correspond to the point of Δ which lies on the normal segment through P, we obtain a (1,1) correspondence between the points of Δ and those of K. We must show that this correspondence is continuous. Suppose that $T_1, T_2, \cdots$ is a sequence of points of K converging to the unique limit point T of K, and let $T_1', T_2', \cdots$ be the corresponding points of Δ. This latter sequence will have one or more limit points which must lie on the normal segment through T. But there can not be more than one such limit point, for otherwise Δ, being a closed set, would contain these points and would thus intersect the normal segment through T at more than one point which can not happen, by XI. Therefore $T_1', T_2', \cdots$ converge to a unique limit point T' which is the point of Δ corresponding to T.

This proves that the correspondence is continuous. The distance between any pair of corresponding points is $< \eta_3$ since $\mu < \eta_3$. Hence XII is completely established.

COROLLARY. *Δ is a simple closed curve.*

By V and XII we have

VIII. *Δ can be mapped $(\eta_1 + \eta_2 + \eta_3)$ on J.*

Since η_1, η_2, η_3 were arbitrary numbers, we can assume that

$$2(\eta_1 + \eta_2) + \eta_3 < \varepsilon_n.$$

ε_n being defined in Theorem 1. Hence (5) may be written

$$|P(x, y)-J(x, y)|<\varepsilon_n$$

and by XIII Δ can be mapped (ε_n) on J. Hence $P(x, y)$ may be taken for the polynomial $P_n(x, y)$ of Theorem 1, which completes the proof of that theorem.

4. **Behavior of $P_n(x, y)$ outside of R'.** Let Q be the square region $0<x, y<1$. The lines $x=\frac{1}{3}$, $x=\frac{2}{3}$, $y=\frac{1}{3}$, $y=\frac{2}{3}$ divide Q into nine square regions of equal area. Call the central region S_0 and the others $S_1, S_2, \cdots, S_8$.

Let c be a simple closed curve in S_0 consisting of a finite number of analytic arcs and let $f(x, y)$ be a function defined and continuous at every finite point, equal to zero on c, >0 exterior to c, and <0 interior to c. Call C the region interior to c and let M be the maximum value of $-f(x, y)$ in C'. Assume finally that for every point not in S_0,

$$f(x, y)>M. \tag{8}$$

LEMMA. *The inequality*

$$\left(\frac{k_{2n}}{2}\right)^2\iint_Q f(u, v)\ [1-(u-x)^2]^{2n}\ [1-(v-y)^2]^{2n}\,du\,dv>0 \tag{9}$$

holds for every finite point not in S_0', and for every value of $n>0$.

We have

$$\iint_Q=\iint_C+\iint_{Q-C}.$$

Obviously, for any finite point not in S_0' and for any positive n,

$$\iint_C<0, \quad \iint_{Q-C}>0,$$

from the definition of $f(x, y)$ and the fact that the other factor of the integrand is never negative. Hence $\iint_Q>0$ if

$$\iint_{Q-C}>-\iint_C$$

or, *a fortiori*, if

$$\iint_{S_i}>-\iint_C, \tag{10}$$

S being any one of the regions $S_1, \cdots, S_8$, since $\iint_{Q-C-S_i} > 0$. Hence it is sufficient to show that for each finite (x, y) not in S_0', the inequality (10) holds for at least one S_i regardless of the value of n.

But this is true because (a) the area of each S_i is greater than the area of C; (b) if (u, v) is any point of C and (u_i, v_i) any point of S_i, then

$$f(u_i, v_i) > M > -f(u, v)$$

by (8) and the definition of M; (c) if (x, y) is any finite point not in S_0', then for each n there is at least one S_i such that

$$[1-(u_i-x)^2]^{2n}\,[1-(v_i-y)^2]^{2n} > [1-(u-x)^2]^{2n}\,[1-(v-y)^2]^{2n}$$

(u, v) and (u_i, v_i) having the same meaning as in (b). To prove (c) we observe first that for a fixed x not in the interval (0, 1) the function

$$k(u) = [1-(u-x)^2]^{2n}$$

has no maximum in the interval (0, 1) but is either monotone or has a single minimum. For

$$\frac{dk}{du} = -4n(u-x)\,[1-(u-x)^2]^{2n-1}.$$

Since $u \neq x$ in the interval (0, 1) $\frac{dk}{du}$ can only be zero if at all, for $u = x+1$ or $u = x-1$. One, but not both, of these values may fall in the interval (0, 1). For that value, $k(u) = 0$ and hence that value is a minimum since $k(u)$ is never negative.

From this it is clear that for any x not in (0, 1) $k(u)$ is greater point for point in one at least of the intervals $(0, \frac{1}{3})$ $(\frac{2}{3}, 1)$ (end-points excepted) than it is in $(\frac{1}{3}, \frac{2}{3})$. (c) follows easily.

THEOREM 2. *The $P_n(x, y)$ of Theorem 1 can be so chosen that for each n, $P_n(x, y) > 0$ at every finite point not in R'.*

Proof. Let K_n be the rounded-off polygon used in obtaining $P_n(x, y)$ (§ 3) and $K_n(x, y)$ the corresponding distance function. Let us now choose the rectangular region R so large that on its boundary

$$K_n(x, y) > N_n \qquad (n = 1, 2, \cdots)$$

where N_n is the maximum value of $-K_n(x, y)$ in the interior region of K_n.*

* It is obvious that R can be so chosen if we recall that the functions $K_n(x, y)$ converge to the function $J(x, y)$.

Let us recall that $P_n(x, y)$ is a polynomial which approximates $K_n(x, y)$ uniformly in R' to a pre-determined degree of accuracy, while the first derivatives of $P_n(x, y)$ approximate those of $K_n(x, y)$ in the neighborhood of K_n. According to § 2, $P_n(x, y)$ can be obtained by first transforming R linearly into a region which we can take to be S_0, lying in Q. $K_n(x, y)$ is thereby transformed into $\bar{K}_n(x, y)$ which can be approximated in S_0' as closely as is necessary by a certain polynomial $\bar{P}_n(x, y)$ of the type (9) whose derivatives approximate those of $\bar{K}_n(x, y)$. On transforming back, the desired $P_n(x, y)$ is obtained.

On the boundary of S_0 and hence also at every point not in S_0', we will have

$$\bar{K}_n(x, y) > N_n.$$

Hence all the conditions of the Lemma are satisfied and it follows that

$$\bar{P}_n(x, y) > 0$$

for every finite point not in S_0', and hence

$$P_n(x, y) > 0$$

for every finite point not in R'.

Since $P(x, y)$ is zero in R' only on the oval approximating J, we have the

COROLLARY. *The $P_n(x, y)$ of Theorem* 1 *can be so chosen that for each n, the corresponding approximating oval is the only real branch of the curve $P_n(x, y) = 0$.*

5. **Approximation by unicursal curves.** THEOREM 3. *Given an arbitrary positive number ϵ, there exists a unicursal algebraic curve whose only real branch is a non-singular oval which can be mapped (ϵ) on J.*

Proof. We shall let K, η_1, η_2, retain the meanings already given them in § 3. Let a parametric representation of K be given by

$$x = F(\lambda), \quad y = G(\lambda)$$

where $\lambda = \dfrac{2\pi}{S} s$, S being the total length of arc of K, and s the length of arc measured in a fixed sense from a fixed point of K. We shall assume that F and G are defined for all finite values of λ. Hence

(a) F and G are continuous for all finite values of λ and periodic, each possessing the period 2π.

Let $\theta(\lambda)$ be the angle which a tangent to K at the point $(F(\lambda), G(\lambda))$ makes with the x-axis. Owing to the rounded nature of K, $\theta(\lambda)$ is continuous for all values of λ while $\sin\theta(\lambda)$ and $\cos\theta(\lambda)$ considered as functions of λ each possess the period 2π.

We have

$$(11)\qquad \begin{aligned} F'(\lambda) &= \frac{dx}{d\lambda} = \frac{S}{2\pi}\frac{dx}{ds} = \frac{S}{2\pi}\cos\theta(\lambda), \\ G'(\lambda) &= \frac{dy}{d\lambda} = \frac{S}{2\pi}\frac{dy}{ds} = \frac{S}{2\pi}\sin\theta(\lambda). \end{aligned}$$

Hence

(b) $F'(\lambda)$ and $G'(\lambda)$ are continuous and periodic, each of period 2π.

(c) F, G, F', G', are bounded and each possesses only a finite number of maxima and minima when λ varies over any interval of length 2π.

Now let α be a number such that

$$(12)\qquad 0 < \alpha < \frac{S}{8\pi},$$

and let $\delta > 0$ be such that, if λ_i and λ_j are any two values of λ satisfying the inequality

$$|\lambda_i - \lambda_j| < \delta,$$

then

$$(13)\qquad \begin{aligned} |F'(\lambda_i) - F'(\lambda_j)| &< \alpha, \\ |G'(\lambda_i) - G'(\lambda_j)| &< \alpha. \end{aligned}$$

Having thus defined δ, let η_4 be an arbitrary positive number and let ξ be such that

$$(14)\qquad 0 < \xi < \eta_4$$

and such that, if P_1 and P_2 are any two points of K with $\overline{P_1P_2} < 2\xi$, then

$$|\lambda_1 - \lambda_2| < \delta$$

where λ_1 and λ_2 are suitably* chosen values of λ corresponding to P_1 and P_2. The existence of ξ follows easily from continuity considerations and the fact that K is without multiple points.

By virtue of (a), (b), and (c) we can approximate the functions F and G uniformly by Fourier Sums whose derivatives approximate F' and G' respectively. More precisely, there exist functions $\varphi(\lambda)$ and $\psi(\lambda)$ defined by

* i. e. with regard to addition of suitable multiples of 2π.

$$\varphi(\lambda) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos n\lambda + b_n \sin n\lambda,$$

(15)

$$\psi(\lambda) = \frac{c_0}{2} + \sum_{n=1}^{N} c_n \cos n\lambda + d_n \sin n\lambda,$$

where N is sufficiently large and the a_n, b_n, c_n, d_n are Fourier coefficients, such that

$$|F(\lambda) - \varphi(\lambda)| < \frac{\xi}{\sqrt{2}},$$

(16)

$$|G(\lambda) - \psi(\lambda)| < \frac{\xi}{\sqrt{2}},$$

and

$$|F'(\lambda) - \varphi'(\lambda)| < \alpha,$$

(17)

$$|G'(\lambda) - \psi'(\lambda)| < \alpha,$$

for all values of λ.

I. *The curve Ω represented by the equations*

$$(\Omega) \qquad x = \varphi(\lambda), \qquad y = \psi(\lambda)$$

can be mapped (η_4) on K and is therefore a simple closed curve.

A continuous correspondence between the points of K and those of Ω may be defined by letting the point of Ω with parameter value λ correspond to the point of K having the same parameter value. The distance $\Delta(\lambda)$ between corresponding points is given by

$$\Delta^2(\lambda) = [F(\lambda) - \varphi(\lambda)]^2 + [G(\lambda) - \psi(\lambda)]^2$$

and by (16) and (14)

$$\Delta(\lambda) < \xi < \eta_4 \tag{18}$$

for all values of λ.

It remains to show that the correspondence established between Ω and K is (1, 1). Let us consider as identical, values of λ differing by integer multiples of 2π. Then to each point of K there corresponds only one value of λ and hence one point of Ω. We must then show that to each point of Ω there corresponds only one value of λ and hence only one point of K. To prove this suppose that to the point P_0 of Ω there correspond two values of λ, λ_1 and λ_2. Let Q_1 and Q_2 be the two points of K corresponding to these parameter values. By (18)

$$\overline{Q_1 P_0} < \xi, \qquad \overline{Q_2 P_0} < \xi$$

and hence $\overline{Q_1 Q_2} < 2\xi$. Hence by choice of ξ

$$|\lambda_1 - \lambda_2| < \delta .$$

Now by assumption

$$\varphi(\lambda_1) = \varphi(\lambda_2), \qquad \psi(\lambda_1) = \psi(\lambda_2) .$$

Hence there must exist by Rolle's Theorem numbers λ_3 and λ_4 where

$$\lambda_1 < \lambda_3 < \lambda_2$$
$$\lambda_1 < \lambda_4 < \lambda_2$$

such that

$$\varphi'(\lambda_3) = 0, \qquad \psi'(\lambda_4) = 0 .$$

Hence by (17)

(19) $$F'(\lambda_3) < \alpha, \qquad G'(\lambda_4) < \alpha .$$

Since $|\lambda_3 - \lambda_4| < \delta$, we have by (13)

$$|F'(\lambda_3) - F'(\lambda_4)| < \alpha$$

and hence by (19, i)

(20) $$F'(\lambda_4) < 2\alpha .$$

Combining (11) with (19, ii) and (20)

$$|\cos\theta(\lambda_4)| < \frac{2\pi}{S} 2\alpha, \qquad |\sin\theta(\lambda_4)| < \frac{2\pi}{S}\alpha .$$

Hence by (12)

$$|\cos\theta(\lambda_4)| < \frac{1}{2}, \qquad |\sin\theta(\lambda_4)| < \frac{1}{4} .$$

But this is absurd, and hence to no point of Ω can there correspond two values of λ. This establishes I.

As a consequence of I together with V of § 3, we have

II. *Ω can be mapped $(\eta_1 + \eta_2 + \eta_4)$ on J.*

Since η_1, η_2, η_4 are arbitrary positive numbers, we can assume that

$$\eta_1 + \eta_2 + \eta_4 < \varepsilon ,$$

where ε is given in Theorem 3. In order now to complete the proof of this theorem, it remains to show that Ω is the only real branch of a unicursal curve.

Since $\sin n\lambda$ and $\cos n\lambda$ are expressible as rational functions of $\tan\frac{\lambda}{2}$, we can, by making the substitution

$$u = \tan\frac{\lambda}{2}$$

reduce the equations (Ω) to

$$x = R_1(u), \qquad y = R_2(u), \tag{21}$$

where R_1 and R_2 are rational functions with real coefficients. As λ varies from $-\pi$ to $+\pi$, u varies from $-\infty$ to $+\infty$ and the corresponding point of Ω describes Ω exactly once. Hence, with the exception of the point which corresponds to $\pm\infty$, there is a (1, 1) correspondence between the points of Ω and the real values of u. Hence Ω is a complete real branch of the unicursal curve which is represented by (21) when u is allowed to assume all complex values. By the Theorem of Harnack a curve of genus p can have at most $p+1$ real branches.* Hence Ω is the *only* real branch of U, and Theorem 3 is thus completely established.

6. **Discussion of results for curves.** The work of § 5 does not preclude the possibility of the unicursal curve U having real isolated points. It is conceivable for example, that on demanding that Ω approximate J more and more closely (corresponding, say, to the sequence $\varepsilon_1, \varepsilon_2, \cdots$ of Theorem 1) the set of real isolated points tends to become everywhere dense in some region containing J. Suppose that for each ε_n the corresponding U has a polynomial representation $S_n(x, y) = 0$. The polynomials $S_n(x, y)$ would certainly not converge uniformly in any region containing J to a function of the type $J(x, y)$.

On the other hand, we have obtained no information concerning the genus of the curves $P_n(x, y) = 0$ of Theorem 1. In other words, we can not conclude from our methods that Theorems 1 and 3 can be combined.

7. **Approximation of manifolds in n-Space.** We extend Theorem 1 to $(n-1)$-dimensional manifolds† M^{n-1} in a number space S^n, admitting of a local parametric representation of the form

$$x^i = \varphi^i(u^1, \cdots, u^{n-1}), \qquad (i = 1, 2, \cdots, n).$$

The points $(u^1, \cdots, u^{n-1})$ describe an $(n-1)$-cell C^{n-1} in which the φ^i together with their first derivatives are continuous, the φ^i being single-valued. At no point of C^{n-1} do the Jacobians

* See for example, Severi-Löffler: Vorlesungen über Algebraische Geometrie, p. 239. It is implied in this proof that a real branch is any set of real points of the curve whose image on the Riemann Surface is a 1-circuit. Ω obviously satisfies this definition.

† As defined by O. Veblen, Analysis Situs, p. 88.

$$\frac{\partial(x^i, x^j, \cdots, x^k)}{\partial(u^1, u^2, \cdots, u^{n-1})}$$

all vanish. By these assumptions M^{n-1} possesses at each point a unique normal which varies continuously with the point.

By the Jordan-Brouwer Theorem* S^n is separated by M^{n-1} into two connected regions. Hence a distance function $M^{n-1}(x^1, \cdots, x^n)$ may be defined with respect to M^{n-1}.

Let R^n be a generalized rectangular region containing M^{n-1}.

THEOREM 4. *Given an $\varepsilon > 0$, there exists a polynomial $A(x^1, \cdots, x^n)$ such that*

$$| M^{n-1}(x^1, \cdots, x^n) - A(x^1, \cdots, x^n) | < \varepsilon$$

in R_n, while the real finite points of the algebraic manifold

$$A(x^1, \cdots, x^n) = 0 \tag{22}$$

constitute a manifold which can be mapped (ε) on M^{n-1}.

$A(x^1, \cdots, x^n)$ may be obtained precisely as we obtained the corresponding polynomial for the rounded-off polygon K, § 3.† The only detail which must be considered is the construction of a region U^n by means of normal segments, analogous to the ring-shaped region U of § 3, in which the first derivatives of $M^{n-1}(x^1, \cdots, x^n)$ are continuous. It will be sufficient to prove the following propositions:

I. For each point P of M^{n-1}, there exists a number $\eta > 0$ and an $(n-1)$-cell B_P on M^{n-1} containing P, such that no two normal segments of length 2η and mid-points distinct points of B_P meet.

Proof. In a small neighborhood V_P of P on M^{n-1} corresponding to $(u^1, \cdots, u^{n-1})$ in a cell C_P^{n-1}, each normal is given a unique sense by bringing a variable, sensed normal into coincidence with it. To each of the points on these normals is associated one or more sets

$$(u^1, \cdots, u^{n-1}, n),$$

where n is the distance, with proper sign, from M^{n-1} to the point along one of the normals through it. The corresponding x-coördinates are given, say, by

$$x^i = \psi^i(u^1, \cdots, u^{n-1}, n), \qquad (u^1, \cdots, u^{n-1}) \text{ in } C_P^{n-1}. \tag{23}$$

* See J. W. Alexander, Proof and Extension of the Jordan-Brouwer Theorem, Trans. Amer. Math. Soc., vol. 23 (1922), p. 333.

† The approximation theorem of § 2 generalizes without difficulty, and an obvious extension of § 4 allows us to choose $A(x^1, \cdots, x^n)$ so that its finite real points constitute but a single manifold.

Now at P, the Jacobian

$$J = \frac{\partial(\psi^1, \psi^2, \cdots, \psi^n)}{\partial(u^1, \cdots, u^{n-1}, n)}$$

is equal to

$$\sum \frac{\partial \psi^i}{\partial n} A^i = \sum \frac{\partial x^i}{dn} A^i = \sum \alpha^i A^i,$$

where the α^i are the direction cosines of the normal at P, and A^i is the cofactor of $\frac{\partial \psi^i}{dn}$ in J. The A^i are proportional to the α^i; and hence

$$\sum \alpha^i A^i = K \sum (\alpha^i)^2, \qquad K \neq 0.$$

Hence at P, $J \neq 0$. But the derivatives $\frac{\partial \psi^i}{\partial u^j}$ are continuous since the $\frac{\partial \varphi^i}{\partial u^j}$ are; also the derivatives $\frac{\partial \psi^i}{\partial n}$ are continuous. Hence there exists an an $(n-1)$-cell D_P contained in C_P^{n-1}, and a number $\eta_1 > 0$ such that for, $(u^1, \cdots, u^{n-1})$ in D_P and $|n| < \eta_1$, $J \neq 0$. Hence for these points (23) possesses a single-valued inverse. Hence if B_P is the image of D_P on M^{n-1}, no two normal segments of length 2η, $\eta < \eta_1$, and distinct mid-points in B_P intersect.

II. A value μ for η exists which serves for the whole manifold M^{n-1}. For if not, we could easily show the existence of a point for which I does not hold. As a corollary:

III. A normal segment of length 2μ has only its mid-point in common with M^{n-1}.

IV. One end-point of a normal segment of length 2μ and variable mid-point remains always interior to M^{n-1}, the other exterior. This follows from III.*

We take for U^n the totality of normal segments of length 2μ.

V. The first derivatives of $M^{n-1}(x^1, \cdots, x^n)$ are continuous in U^n. For at a point P in U^n, $\frac{\partial M^{n-1}}{\partial x^i}$ is the rate of change of the normal distance from P to M^{n-1}. This is easily seen to be equal to the cosine of the angle which the normal makes with the x^i-axis, and hence it is continuous.

8. **More general manifolds.** Theorem 4 obviously holds for any closed manifold H^{n-1} in S^n for which there exists a manifold of type M^{n-1}

* For, if AB is a normal segment with variable mid-point P, then by III PA (P not included) is always on the same side of M^{n-1}; similarly for PB. Now the totality $\{PA\}$ of segments PA plus the totality $\{PB\}$ constitutes a complete neighborhood of M^{n-1}, which consists of two disjoint parts since M^{n-1} is two-sided. Hence $\{PA\}$ is on one side of M^{n-1} and $\{PB\}$ on the other.

upon which it may be mapped (ε). It is not known that the most general H^{n-1} can be thus approximated. We shall indicate briefly however how this may be done for 2 and 3-dimensional polyhedra in S^3 and S^4 respectively, and point out what prevents our method from extending to n dimensions.

Let Π^{n-1} ($n = 3$ or 4) be a non-singular $(n-1)$-dimensional polyhedron* in S^n.

THEOREM 5. *There exists a manifold K^{n-1} of type M^{n-1} (§ 7) which can be mapped (ε) on Π^{n-1}, $n = 3$ or 4.*

Proof. K^{n-1} will be obtained by "rounding off the corners" of Π^{n-1} by fitting in pieces of tangent surfaces. These surfaces will always be generated by an arc of a circle or a portion of a sphere. The resulting manifold will possess a continuously turning tangent plane and will be seen therefore to possess the properties of an M^{n-1}. We shall treat separately the cases $n = 3$ and $n = 4$.

$n = 3$. By introducing new edges when necessary, we may assume that the faces of Π^2 are triangular. Since Π^2 is non-singular, two faces abut on each edge.

Let it be understood once and for all that the upper bound of each radius mentioned in the following is to be so small that the statements which will be made concerning mapping will be true.

Now let P be a variable point of an edge e_1 whose ends are the vertices V_1 and V_2. A plane perpendicular to e_1 at P, intersects the two faces abutting on e_1 in a pair of lines whose angle θ is constant. We may assume $\theta \neq \pi$; for, if $\theta = \pi$, there is no rounding to be done along e_1 and we proceed to another edge. Into θ we fit an arc σ of a tangent circle of radius ϱ (see fig. 3, § 3) and σ can be mapped (ε) on the broken line which it replaces. In the neighborhood of V_1, ϱ, which is a function of P, is to approach zero *linearly* as P approaches V_1; similarly at V_2. Elsewhere on e_1, ϱ is to vary so that $\frac{d\varrho}{d\xi}$, $\xi = V_1 P$, is continuous. The resulting surface S_{e_1} generated by σ, consists near V_1 (also V_2) of a portion of a circular cone with apex V_1. S_{e_1} possesses a continuously turning tangent plane and can be mapped (ε) on the part of Π^2 which it replaces.

We treat the remaining edges in a similar manner, choosing the upper bounds of the radii so small that no two surfaces S_e intersect. We thus obtain a surface Π_1^2 which can be mapped (ε) on Π^2 and which possesses a continuously turning tangent plane except at the vertices.

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* For a definition of polyhedron see O. Veblen, On the Decomposition of n-Space by a Polyhedron, Trans. Amer. Math. Soc. 14 (1913), p. 65. By *non-singular* we shall mean that, if P is any point of Π^{n-1}, the points of Π^{n-1} interior to and on an $(n-1)$-sphere of sufficiently small radius and center P constitute an $(n-1)$-cell and its boundary.

Now let a sphere O of radius τ be placed about V_1 for center. For τ sufficiently small the points of Π^2 interior to O constitute a 2-cell. It is easily seen that the same statement holds for Π_1^2, the 2-cell being, say, c^2. We further take τ so small that O cuts the surfaces S_e adjacent to V_1 only in the portions where they are conical. With τ so chosen, we see that O is *orthogonal* to Π_1^2, cutting it in a simple rounded-off polygon L. We replace c^2 by one of the two 2-cells on O, call it d^2, which L bounds. c^2 can be mapped (ε) on d^2 and this is to be done in such a way that L considered as the boundary of c^2 corresponds to itself considered as the boundary of d^2.

There is no difficulty now in rounding off the corner along L which d^2 makes with $\Pi_1^2 - c^2$, by fitting a 2-dimensional collar which can be mapped (ε) on the part of Π_1^2 which it replaces. The collar is generated precisely like a surface S_e but with a generating arc of constant radius.

Treating the remaining vertices of Π_1^2 in like manner, we have the desired K^2.

$n = 4$. We merely outline the argument here which follows closely that of the previous case.

By introducing new 2-faces when necessary, we may assume that all the 3-faces of Π^3 are tetrahedral, and hence all 2-faces triangular. Two 3-faces abut at each 2-face.

Let P be a variable point of a 2-face F_1^2 at which the 3-faces F_1^3 and F_2^3 abut. An S^2 perpendicular to F_1^3 and F_2^3 at P cuts them in a pair of lines forming a constant angle which we may assume to be $\neq \pi$. We round off this angle by an arc of radius $r = \varphi(P)$ where $\varphi(P)$, a single-valued function of P, has the property that it approaches zero *linearly* as P approaches the boundary B of F_1^2 along a line either perpendicular to an F^1 on B or through an F^0 on B. Elsewhere on F_1^2, $\frac{\partial r}{\partial \xi}$ and $\frac{\partial r}{\partial \eta}$ are to be continuous, where ξ and η are the coördinates of P with regard to a cartesian system set up in the plane of F_1^2. Finally $\varphi(P)$, always positive, is to have an upper bound small as we please.*

Having treated all 2-faces in like manner, we obtain a manifold Π_1^3 having the desired properties except along its edges and vertices. An S^3 perpendicular to an edge F^1 at a variable point Q, intersects Π_1^3 near Q in a set of points whose structure is precisely that of a neighborhood of a vertex

* Such a function may be defined by means of a tetrahedral surface T erected on F_1^2 for its base, the fourth vertex D having its orthogonal projection on F_1^2. The edges meeting at D are rounded off by the methods for obtaining Π_1^2. Then D is rounded off by a suitable sphere O and collar. In order that the function shall be single-valued, it will be necessary that the cell of O which replaces the neighborhood of D shall be the one interior to T.

of a Π_1^2, and is accordingly rounded off by means of a sphere of radius s and a 2-dimensional collar the radius of whose generating arc is t. s and t are to vary with Q precisely as ϱ varied with P in the construction of Π_1^2. The remaining edges are similarly treated.

We now have a manifold Π_2^3 which has the desired properties except at the vertices. Owing to the linear nature of the variation of the various radii in the neighborhood of a vertex, it is clear that the structure of Π_2^3 near a vertex F^0 is that of a sheaf of lines through F^0. Hence a hypersphere O^3 of small radius and center F^0 is *orthogonal* to Π_2^3 intersecting it in a closed surface L^2 without corners. The points of Π_2^3 interior to and on O^3 constitute a 3-cell c^3 and its boundary. Hence L^2 is of genus zero and hence, by a theorem due to J. W. Alexander,* L^2 bounds two 3-cells on O^3, one of which we call d^3. We replace c^3 by d^3 and round off the corner between d^3 and $\Pi_2^3 - c^3$ by fitting in 3-dimensional collar of obvious construction.

When the remaining vertices of Π_2^3 are similarly treated, the desired K^3 will be obtained.

The above method demands for its extension to any n that the theorem of Alexander just referred to hold for spaces of any dimensionality, and this is still an open question.

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* On the Subdivision of 3-Space by a Polyhedron, Proc. Nat. Acad. Sci., vol. 1 (1924), p. 6.

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